

Precision of Phase Correlation for Speckle Tracking

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Abstract—The phase correlation method is widely used to estimate image offsets and is generally preferred over the cross correlation method due to its perceived high precision derived from phenomenological observations of correlation peaks. For coherent imaging systems, the inherent speckle in the images carries valuable information that can be exploited for offset estimation, a technique known as the speckle tracking. Assuming the canonical signal model for speckle, we derive the offset estimation precision for both the coherent phase correlation (CPC) and the incoherent phase correlation (IPC). The IPC precision is derived under large number of independent samples. It is shown that the CPC exhibits lower precision than the coherent cross correlation (CCC). The IPC similarly yields lower precision than the incoherent cross correlation (ICC), but the difference between them is marginal. The mechanism underlying the reduced precision of the CPC and IPC, despite their sharper correlation peaks, is explained. Furthermore, we define a generalized correlation family, termed the fractional exponent correlation. The phase correlation and the cross correlation are merely special cases of the fractional exponent correlation, and by tuning the exponent of the spectral magnitude of the fractional exponent correlation function, an avenue for improving offset estimation precision is suggested.

Index Terms—cross correlation, offset estimation, phase correlation, speckle.

I. INTRODUCTION

Estimating image offsets is a prerequisite for multi-image analysis. A number of methods have been developed to estimate image offsets with varying efficiency, robustness, and precision [1], [2], [3], [4]. Precision can be critical for applications such as Synthetic Aperture Radar (SAR) Interferometry (InSAR) and ground motion measurements using SAR or optical remote sensing images, which generally require subpixel precision or higher.

The time-domain cross correlation is one of the most intuitive approaches for estimating offsets with high precision. The time-domain offset corresponds to a linear phase in the frequency domain, which gives rise to the approach known as the phase correlation [5], [6]. It is generally concluded that phase correlation is superior to cross correlation, because its correlation peak is sharper and it is less dependent on image intensity and content [6], [7]. Therefore, phase correlation is

preferred and has been widely applied to image coregistration and motion measurement for images acquired by sensors based on various physical principles [6], [8], [9], [10], [11], [12], [13].

The general conclusion that phase correlation achieves a higher precision has been drawn primarily from phenomenological observations, rather than theoretical analysis. It is challenging to mathematically model the image spatial variations (hereafter referred to as "features"), which are manifestations of the physical properties of imaged objects through the imaging system. For coherent imaging systems, including coherent radar, optical, and acoustic systems, an inherent characteristic of the images is the speckle, arising from wave interference. The technique that utilizes speckle to estimate image offset is commonly referred to as the speckle tracking. Speckle tracking has significantly contributed to the subpixel precision required for InSAR and ground motion measurement when there is sufficient coherence between SAR images.

Fully developed speckle is typically modeled as a zero-mean complex circular Gaussian (CCG) random variable [14], which provides an opportunity to derive the offset estimation precision for the correlation methods. Under this model, the theoretical precisions of coherent and incoherent cross correlation have been derived [15], [16]. However, the theoretical precision of phase correlation remains unknown.

In this work, we derive the theoretical precision of phase correlation under the assumption of a zero-mean CCG model and validate the theoretical findings through simulations and observations in real-data experiments.

II. PRECISION OF PHASE CORRELATION FOR ZERO-MEAN CCG SIGNALS

A. Signal Model

Assume that the signal is critically sampled. Each sample represents a resolution cell that can be modeled as fully developed speckle following a zero-mean CCG distribution [14]. The samples are independent and identically distributed (i.i.d.).

Let the samples of the reference and secondary signals be denoted by x_n and y_n , respectively, where n is the sample index. Without loss of generality, the signal power is normalized to unity, i.e., $E[|x_n|^2] = E[|y_n|^2] = 1$. Then, the complex coherence between x_n and y_n can be simply expressed as $\gamma = E[x_n y_n^*] = |\gamma| e^{j\psi_0}$, where ψ_0 is the expected interferometric phase. For simplicity, ψ_0 is assumed to be zero; hence, γ is a nonnegative real quantity.

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The probability density function (PDF) of the interferometric phase ψ is given by [14]

$$p(\psi; \gamma) = \frac{1 - \gamma^2}{2\pi} \cdot \frac{1 + \gamma \cos \psi \cdot \arccos(-\gamma \cos \psi) / \sqrt{1 - \gamma^2 \cos^2 \psi}}{1 - \gamma^2 \cos^2 \psi}. \quad (1)$$

Because $\psi_0 = 0$, ψ also represents the interferometric phase noise. This distribution possesses the following properties that are essential for the subsequent derivation of offset estimation precision:

1) Symmetry:

$$p(\psi; \gamma) = p(-\psi; \gamma).$$

2) Sine expectation:

$$E(\sin \psi) = \int_{-\pi}^{\pi} p(\psi; \gamma) \cdot \sin \psi d\psi = 0.$$

3) Sine variance:

$$S(\gamma) \equiv E(\sin^2 \psi) = \frac{\gamma^2 - 1}{2\gamma^2} \ln(1 - \gamma^2).$$

4) Cosine expectation:

$$C(\gamma) \equiv E(\cos \psi) = \frac{E(\gamma^2) - (1 - \gamma^2)K(\gamma^2)}{\gamma}.$$

where $K(\cdot)$ and $E(\cdot)$ are the complete elliptic integrals of the first and second kinds, respectively, with parameter γ^2 .

The detailed derivations of these properties are provided in the Appendix A.

The normalized Discrete Fourier Transform (DFT) with a scaling factor of $1/\sqrt{N}$ is a linear unitary operator. It follows that the coefficients of the normalized DFT of the time-domain samples remain i.i.d., each following the zero-mean CCG distribution identical to that in the time domain [17]. For the standard DFT, the standard deviation of each coefficient is amplified by $\sqrt{N}$. The DFT coefficients are denoted by $X_k = \mathcal{F}\{x_n\}$ and $Y_k = \mathcal{F}\{y_n\}$. As proved in Appendix C-A, the complex coherence between X_k and Y_k is also equal to γ , regardless of whether the normalized or standard DFT is used.

Prior to intensity detection, the complex signal must be oversampled by a factor of 2, yielding $N_2 = 2N$ samples, because intensity detection doubles the signal bandwidth [18]. Let x'_n and y'_n denote the samples of the twofold-oversampled reference and secondary signals, respectively. Their intensities are given by $u_n = |x'_n|^2$ and $v_n = |y'_n|^2$. Then, $E[u_n] = E[v_n] = 1$, and $\text{Var}(u_n) = \text{Var}(v_n) = 1$. The corresponding DFT coefficients are denoted by $U_k = \mathcal{F}\{u_n\}$ and $V_k = \mathcal{F}\{v_n\}$, which are equivalent to the outputs of spectral autocorrelation. Further, define the phase-only intensity cross-spectrum phasor $q_k = U_k V_k^* / |U_k V_k^*| = e^{j\psi_k}$. Since u_n and v_n are real, $q_{-k} = q_k^*$.

From this point forward, unless otherwise stated, we adopt the following conventions to facilitate the derivations:

1) The number of samples in each original signal, N , is even.

2) The index sets for the original and the twofold-oversampled signals are defined as

$$\mathcal{J}_N = \left\{-\frac{N}{2} + 1, \dots, \frac{N}{2}\right\}, \quad \mathcal{J}_{2N} = \{-N + 1, \dots, N\}. \quad (2)$$

3) The N -point standard DFT pairs for the original, critically sampled complex signals x_n and y_n are given by

$$X_k = \sum_{n \in \mathcal{J}_N} x_n e^{-j2\pi kn/N}, \quad x_n = \frac{1}{N} \sum_{k \in \mathcal{J}_N} X_k e^{j2\pi kn/N},$$

$$Y_k = \sum_{n \in \mathcal{J}_N} y_n e^{-j2\pi kn/N}, \quad y_n = \frac{1}{N} \sum_{k \in \mathcal{J}_N} Y_k e^{j2\pi kn/N}. \quad (3)$$

4) The twofold-oversampled signal is obtained by padding both ends of X_k or Y_k with $N/2$ zero-valued samples.

B. Precision of Phase Correlation for Complex Signals

Phase correlation utilizes the shift property of the Fourier transform to estimate the offset. The phase correlation function in the time domain based on the standard DFT can be defined as [1]:

$$c_{\text{CPC}}(\xi) = \frac{1}{N} \sum_{k \in \mathcal{J}_N} \frac{X_k Y_k^*}{|X_k Y_k^*|} \cdot e^{j\omega_k \xi}$$

$$= \frac{1}{N} \sum_{k \in \mathcal{J}_N} e^{j(-\omega_k \xi_0 + \psi_k)} \cdot e^{j\omega_k \xi} \quad (4)$$

where $\omega_k = 2\pi k/N$, $k \in \mathcal{J}_N$, ξ_0 is the true offset, and ψ_k is the interferometric phase noise at frequency bin k . $c_{\text{CPC}}(\xi)$ is the standard IDFT coefficient, but with the discrete index n replaced by a continuous variable ξ , which represents the candidate offset. By normalizing the cross-power spectrum $X_k Y_k^*$, all frequency components contribute equally to $c_{\text{CPC}}(\xi)$, regardless of their signal-to-noise ratio (SNR). This correlation method is referred to as the coherent phase correlation (CPC). Without the denominator $|X_k Y_k^*|$, (4) becomes the phase correlation function of the coherent cross correlation (CCC).

By finding the maximum of $|c_{\text{CPC}}(\xi)|$, the offset can be estimated as

$$\hat{\xi} = \arg \max_{\xi} \{|c_{\text{CPC}}(\xi)|\}. \quad (5)$$

Without loss of generality, we set the true offset $\xi_0 = 0$. Then, (4) becomes

$$c_{\text{CPC}}(\xi) = \frac{1}{N} \sum_{k \in \mathcal{J}_N} e^{j(\psi_k + \omega_k \xi)}. \quad (6)$$

The phase correlation function attains its true maximum at $\xi = \xi_0 = 0$ in the absence of noise.

The variance of $\hat{\xi}$ can be derived via a first-order Taylor expansion of the derivative of the correlation function around the true peak position [19], $\xi = 0$

$$\sigma_{\text{CPC}}^2 = \frac{\text{Var} \left[\left. \frac{d}{d\xi} |c_{\text{CPC}}(\xi)| \right|_{\xi=0} \right]}{\left[E \left[\left. \frac{d^2}{d\xi^2} |c_{\text{CPC}}(\xi)| \right|_{\xi=0} \right] \right]^2}. \quad (7)$$

The derivatives of the phase correlation function at $\xi = 0$ are given by

$$\begin{aligned} \left. \frac{d}{d\xi} |c_{\text{CPC}}(\xi)| \right|_{\xi=0} &\approx \text{Re}\{c'_{\text{CPC}}(0)\} \\ &= \text{Re} \left\{ \frac{1}{N} \sum_{k \in \mathcal{I}_N} j\omega_k e^{j\psi_k} \right\} = -\frac{1}{N} \sum_{k \in \mathcal{I}_N} \omega_k \sin \psi_k, \quad (8) \end{aligned}$$

$$\begin{aligned} \left. \frac{d^2}{d\xi^2} |c_{\text{CPC}}(\xi)| \right|_{\xi=0} &\approx \text{Re}\{c''_{\text{CPC}}(0)\} \\ &= \text{Re} \left\{ -\frac{1}{N} \sum_{k \in \mathcal{I}_N} \omega_k^2 e^{j\psi_k} \right\} = -\frac{1}{N} \sum_{k \in \mathcal{I}_N} \omega_k^2 \cos \psi_k \quad (9) \end{aligned}$$

where the approximations hold because the mean peak is real and positive.

The variance of the first derivative can be calculated as

$$\begin{aligned} &\text{Var} \left[\left. \frac{d}{d\xi} |c_{\text{CPC}}(\xi)| \right|_{\xi=0} \right] \\ &= E \left[\left(\left. \frac{d}{d\xi} |c_{\text{CPC}}(\xi)| \right|_{\xi=0} \right)^2 \right] - \left(E \left[\left. \frac{d}{d\xi} |c_{\text{CPC}}(\xi)| \right|_{\xi=0} \right] \right)^2 \\ &= E \left[\left(-\frac{1}{N} \sum_{k \in \mathcal{I}_N} \omega_k \sin \psi_k \right)^2 \right] - \left[-\frac{1}{N} \sum_{k \in \mathcal{I}_N} \omega_k E(\sin \psi_k) \right]^2 \\ &= \frac{1}{N^2} \sum_{k_1 \in \mathcal{I}_N} \sum_{k_2 \in \mathcal{I}_N} \omega_{k_1} \omega_{k_2} E(\sin \psi_{k_1} \sin \psi_{k_2}) \\ &= \frac{1}{N^2} \sum_{k \in \mathcal{I}_N} \omega_k^2 E(\sin^2 \psi_k) \\ &= \frac{1}{N^2} \sum_{k \in \mathcal{I}_N} \omega_k^2 S(\gamma). \quad (10) \end{aligned}$$

In deriving the third equality, Property 2) of the PDF of the interferometric phase has been used. The fourth equality is obtained by further applying Property 2) and leveraging the fact that the interferometric phase samples are statistically independent with identical coherence, i.e., $\gamma_k \equiv \gamma$.

The expectation of the second derivative can be simply obtained by applying Property 4)

$$\begin{aligned} E \left[\left. \frac{d^2}{d\xi^2} |c_{\text{CPC}}(\xi)| \right|_{\xi=0} \right] &= -\frac{1}{N} \sum_{k \in \mathcal{I}_N} \omega_k^2 E(\cos \psi_k) \\ &= -\frac{1}{N} \sum_{k \in \mathcal{I}_N} \omega_k^2 C(\gamma). \quad (11) \end{aligned}$$

The sum over ω_k^2 can be calculated as follows

$$\sum_k \omega_k^2 = \frac{4\pi^2}{N^2} \sum_{k \in \mathcal{I}_N} k^2 = \frac{(N^2 + 2)\pi^2}{3N}. \quad (12)$$

Substituting these results into (7), the closed-form precision expression for the CPC can be obtained

$$\begin{aligned} \sigma_{\text{CPC}}^2 &= \frac{\text{Var} \left[\left. \frac{d}{d\xi} |c_{\text{CPC}}(\xi)| \right|_{\xi=0} \right]}{\left[E \left[\left. \frac{d^2}{d\xi^2} |c_{\text{CPC}}(\xi)| \right|_{\xi=0} \right] \right]^2} \\ &= \frac{\frac{1}{N^2} S(\gamma) \sum_{k \in \mathcal{I}_N} \omega_k^2}{\left[-\frac{1}{N} C(\gamma) \sum_{k \in \mathcal{I}_N} \omega_k^2 \right]^2} \\ &= \frac{\frac{\gamma^2 - 1}{2\gamma^2} \ln(1 - \gamma^2)}{\frac{(N^2 + 2)\pi^2}{3N} \cdot \left[\frac{E(\gamma^2) - (1 - \gamma^2)K(\gamma^2)}{\gamma} \right]^2} \\ &= \frac{3N}{2(N^2 + 2)} \cdot \frac{(\gamma^2 - 1)\ln(1 - \gamma^2)}{\pi^2 [E(\gamma^2) - (1 - \gamma^2)K(\gamma^2)]^2} \quad (13) \end{aligned}$$

where $K(\cdot)$ and $E(\cdot)$ are the complete elliptic integrals of the first and second kinds, respectively. Several observations follow from (13). First, the variance σ_{CPC}^2 scales as $1/N$, indicating that increasing the number of samples improves offset estimation precision. Second, the coherence-dependent factors are highly nonlinear: As $\gamma \rightarrow 1$, the numerator term $(\gamma^2 - 1)\ln(1 - \gamma^2)$ approaches zero, driving the error to zero; as $\gamma \rightarrow 0$, the elliptic integral combination in the denominator approaches zero faster, causing the error to diverge.

For comparison, the variance for the CCC is given by [16]

$$\sigma_{\text{CCC}}^2 = \frac{3}{2N} \cdot \frac{1 - \gamma^2}{\pi^2 \gamma^2}. \quad (14)$$

C. Precision of Phase Correlation for Intensity Signals

In practice, offset estimation bias arises from the spatial variations of the expected interferometric phase between the two signals when complex signals are used for correlation. For example, such a phase can be induced by atmospheric refraction and ground motions in InSAR. Since it is often challenging to eliminate the expected interferometric phase, signal intensity is generally preferred for correlation.

The phase correlation function, with respect to the twofold-oversampled coordinate ξ , for intensity signals in the time domain based on the standard DFT can be defined as [1]:

$$\begin{aligned} c_{\text{IPC}}(\xi) &= \frac{1}{2N} \sum_{k \in \mathcal{I}_{2N} \setminus \{N\}} \frac{U_k V_k^*}{|U_k V_k^*|} \cdot e^{j\omega_k \xi} \\ &= \frac{1}{2N} \sum_{k \in \mathcal{I}_{2N} \setminus \{N\}} e^{j(\psi_k + \omega_k \xi)} \quad (15) \end{aligned}$$

where $\omega_k = \pi k/N$, and $k \in \mathcal{I}_{2N} \setminus \{N\}$. Under the adopted zero-padding convention, both U_k and V_k evaluate to zero at $k = N$; thus, this frequency bin should be excluded from the computation of (15). This correlation method is referred to as the incoherent phase correlation (IPC). Without the denominator $|U_k V_k^*|$, (15) becomes the phase correlation function of the incoherent cross correlation (ICC). As in the case of complex signal phase correlation, we have assumed the true offset $\xi_0 = 0$.

Pairing the conjugate frequency bins $-k$ and k leads to a real-valued phase correlation function:

$$\begin{aligned} c_{\text{IPC}}(\xi) &= \frac{1}{2N} \sum_{k=-N+1}^{N-1} e^{j(\psi_k + \omega_k \xi)} \\ &= \frac{1}{2N} \sum_{k=-N+1}^{-1} e^{j(\psi_k + \omega_k \xi)} + \frac{1}{2N} \sum_{k=1}^{N-1} e^{j(\psi_k + \omega_k \xi)} + \frac{e^{j\psi_0}}{2N} \\ &= \frac{1}{2N} \sum_{k=1}^{N-1} [e^{j(\psi_k + \omega_k \xi)} + e^{j(-\psi_k - \omega_k \xi)}] + \frac{e^{j\psi_0}}{2N} \\ &= \frac{1}{N} \sum_{k=1}^{N-1} \cos(\omega_k \xi + \psi_k) + \frac{e^{j\psi_0}}{2N}. \end{aligned} \quad (16)$$

It is a real-valued function, and we obtain the variance of $\hat{\xi}$ around the true peak position at $\xi = 0$ via

$$\sigma_{\text{IPC}}^2 = \frac{\text{Var}[c'_{\text{IPC}}(0)]}{|E[c''_{\text{IPC}}(0)]|^2}. \quad (17)$$

From the perspective of spectral autocorrelation, each of U_k and V_k is the sum of $N - k$ complex conjugate products of the DFT coefficients from the oversampled complex samples. For a sufficiently large $N - k$, (U_k, V_k) can be asymptotically approximated by a joint CCG pair with coherence γ_l , as derived in Appendix C-A. According to the properties given in Section II-A, $E[\sin \psi_k] \approx 0$, $E[\cos \psi_k] \approx C(\gamma_l)$, and $E[\sin^2 \psi_k] \approx S(\gamma_l)$, where $C(\cdot)$ and $S(\cdot)$ are defined in Appendix A. Unlike in the case of the CPC, the phases at different frequency bins are not independent.

At $\xi = 0$, the derivatives of the phase correlation function are given by

$$c'_{\text{IPC}}(0) = -\frac{1}{N} \sum_{k=1}^{N-1} \omega_k \sin \psi_k. \quad (18)$$

$$c''_{\text{IPC}}(0) = -\frac{1}{N} \sum_{k=1}^{N-1} \omega_k^2 \cos \psi_k. \quad (19)$$

The variance of the first derivative can be calculated as

$$\text{Var}[c'_{\text{IPC}}(0)] = E[(c'_{\text{IPC}}(0))^2] - (E[c'_{\text{IPC}}(0)])^2$$

$$\begin{aligned} &= \frac{1}{N^2} \sum_{k=1}^{N-1} \sum_{l=1}^{N-1} \omega_k \omega_l E(\sin \psi_k \sin \psi_l) \\ &= \frac{1}{N^2} \sum_{k=1}^{N-1} \omega_k^2 E(\sin^2 \psi_k) \\ &\quad + \frac{2}{N^2} \sum_{1 \leq k < l \leq N-1} \omega_k \omega_l E(\sin \psi_k \sin \psi_l). \end{aligned} \quad (20)$$

According to Appendix C, the cross-frequency term, for $1 \leq k < l \leq N - 1$, is

$$E[\sin \psi_k \sin \psi_l] \approx C^2(\gamma_l) \frac{1 - \gamma_l}{\gamma_l} \frac{\min(k, N - l)}{(N - k)(N - l)}. \quad (21)$$

where $h_{kl} = \min(k, N - l)$. This term originates from the fact that U_k and U_l (V_k and V_l), which are essentially the results of spectral autocorrelation, contain contributions from common DFT coefficients of the original complex signals. Substituting (21) into (20) yields

$$\begin{aligned} \text{Var}[c'_{\text{IPC}}(0)] &\approx \frac{1}{N^2} \sum_{k=1}^{N-1} \omega_k^2 S(\gamma_l) \\ &\quad + \frac{2}{N^2} \sum_{1 \leq k < l \leq N-1} \omega_k \omega_l C^2(\gamma_l) \frac{1 - \gamma_l}{\gamma_l} \frac{\min(k, N - l)}{(N - k)(N - l)}. \end{aligned} \quad (22)$$

The expectation of the second derivative takes the same form as that of the CPC

$$E[c''_{\text{IPC}}(0)] = -\frac{1}{N} \sum_{k=1}^{N-1} \omega_k^2 E(\cos \psi_k) \approx -\frac{1}{N} \sum_{k=1}^{N-1} \omega_k^2 C(\gamma_l). \quad (23)$$

Substituting (22) and (23) into (17) and then mapping back to the original coordinate system, yields

$$\begin{aligned} \sigma_{\text{IPC}}^2 &= \frac{1}{4} \cdot \frac{\text{Var}[c'_{\text{IPC}}(0)]}{|E[c''_{\text{IPC}}(0)]|^2} \\ &\approx \frac{\frac{1}{N^2} \sum_{k=1}^{N-1} \omega_k^2 S(\gamma_l) + \frac{2}{N^2} \sum_{1 \leq k < l \leq N-1} \omega_k \omega_l C^2(\gamma_l) \frac{1 - \gamma_l}{\gamma_l} \frac{\min(k, N - l)}{(N - k)(N - l)}}{4 \left[-\frac{1}{N} \sum_{k=1}^{N-1} \omega_k^2 C(\gamma_l) \right]^2} \\ &= \frac{S(\gamma_l)}{4C^2(\gamma_l)D_N} + \frac{1 - \gamma_l}{2\gamma_l} \frac{G_N}{D_N^2}. \end{aligned} \quad (24)$$

For large N , G_N can be approximated via the Riemann-sum limit

$$G_N = \sum_{1 \leq k < l \leq N-1} \omega_k \omega_l \frac{\min(k, N - l)}{(N - k)(N - l)}$$

$$\begin{aligned}
&= N\pi^2 \sum_{1 \leq k < l \leq N-1} \frac{k l \min(k, N-l)}{N^3 (N-k)(N-l)} \\
&\rightarrow N\pi^2 \int_{0 < x < y < 1} \frac{xy \min(x, 1-y)}{(1-x)(1-y)} dx dy. \quad (25)
\end{aligned}$$

The integral evaluates to $(\pi^2 - 5)/12$; hence,

$$G_N \rightarrow \frac{N\pi^2(\pi^2 - 5)}{12}. \quad (26)$$

For large N , D_N can be approximated as

$$D_N = \sum_{k=1}^{N-1} \omega_k^2 = \frac{(N-1)(2N-1)\pi^2}{6N} \rightarrow \frac{N\pi^2}{3}. \quad (27)$$

Substituting (26) and (27) into (24), we obtain the final closed-form precision expression for the IPC

$$\begin{aligned}
\sigma_{\text{IPC}}^2 &\simeq \frac{3S(\gamma_l)}{4\pi^2 N C^2(\gamma_l)} + \frac{3(\pi^2 - 5)}{8\pi^2 N} \frac{1 - \gamma_l}{\gamma_l} \\
&= \frac{3}{8\pi^2 N} \left\{ \frac{(\gamma^4 - 1) \ln(1 - \gamma^4)}{[E(\gamma^4) - (1 - \gamma^4)K(\gamma^4)]^2} + (\pi^2 - 5) \frac{1 - \gamma^2}{\gamma^2} \right\}. \quad (28)
\end{aligned}$$

(28) consists of two terms that originate from the co-frequency and cross-frequency terms in (22), respectively. Both terms scale with N^{-1} ; therefore, the cross-frequency contribution does not diminish as the number of samples increases.

For comparison, the variance for the ICC is given by [15]

$$\sigma_{\text{ICC}}^2 = \frac{3}{10N} \cdot \frac{2 + 5\gamma^2 - 7\gamma^4}{\pi^2 \gamma^4}. \quad (29)$$

D. Peak Position Variance, Peak Curvature Expectation, and Offset Estimation Variance

In the offset estimation variance given in (17), the numerator, $\text{Var}[c'_{\text{IPC}}(0)]$, denotes the peak position variance (fluctuation), while the denominator, $|E[c''_{\text{IPC}}(0)]|^2$, denotes the squared peak curvature (sharpness) expectation. The offset estimation variance depends on both quantities. The offset estimation precision can be interpreted as the original precision, represented by the peak position variance, scaled by a factor, represented by the inverse of the squared peak curvature expectation.

The sole difference between phase correlation and cross correlation lies in the whitening of the cross-power spectrum, e.g., scaling the cross-power spectrum by $1/|U_k V_k^*|$ in (15). The normalizing or whitening operation in phase correlation assigns a unit magnitude to every frequency bin and therefore effectively preserves the full bandwidth in the frequency domain, which corresponds to a narrow correlation peak (sharper peak, or larger peak curvature) in the time domain. However, it also assigns equal weight to weak and statistically unreliable bins near the spectral edges, which amplifies the noise and hence increases peak position fluctuations. In contrast, cross correlation retains the full spectral magnitude and naturally suppresses these noisy bins,

thereby reducing peak position fluctuations at the expense of a broader peak.

A single realization of the correlation surface, which is typically illustrated in previous studies, only shows the peak sharpness, but does not reflect the peak position fluctuations. Therefore, the widespread conclusion that phase correlation is more precise than cross correlation, drawn solely from the qualitative observation of peak sharpness, is not necessarily true.

Appendix D provides a detailed quantitative comparison of the peak position variance, peak curvature expectation, and offset estimation variance between the ICC and IPC and verifies the derived large- N correlation-surface profiles against Monte Carlo simulations.

III. FRACTIONAL EXPONENT CORRELATION

Because the sole difference between phase correlation and cross correlation lies in the whitening of the cross-power spectrum, a generalized family of correlation functions can be defined by assigning an arbitrary exponent to the magnitude of the cross-power spectrum, which is termed the fractional exponent correlation. For complex signals, the correlation function can be defined following (4)

$$\begin{aligned}
c_{\text{CFEC}}(\xi) &= \frac{1}{N} \sum_{k \in \mathcal{J}_N} A_k^\alpha e^{j(-\omega_k \xi_0 + \psi_k + \omega_k \xi)} \\
&= \frac{1}{N} \sum_{k \in \mathcal{J}_N} A_k^\alpha e^{j(\psi_k + \omega_k \xi)}, \quad \text{for } \xi_0 = 0 \quad (30)
\end{aligned}$$

where $A_k e^{j(-\omega_k \xi_0 + \psi_k)} = X_k Y_k^*$. We refer to this correlation method as the coherent fractional exponent correlation (CFEC). Similarly, for intensity signals, the correlation function can be defined following (15)

$$\begin{aligned}
c_{\text{IFEC}}(\xi) &= \frac{1}{2N} \sum_{k \in \mathcal{J}_{2N} \setminus \{N\}} A_k^\alpha e^{j(-\omega_k \xi_0 + \psi_k + \omega_k \xi)} \\
&= \frac{1}{2N} \sum_{k \in \mathcal{J}_{2N} \setminus \{N\}} A_k^\alpha e^{j(\psi_k + \omega_k \xi)}, \quad \text{for } \xi_0 = 0 \quad (31)
\end{aligned}$$

where $A_k e^{j(-\omega_k \xi_0 + \psi_k)} = U_k V_k^*$. We refer to this correlation method as the incoherent fractional exponent correlation (IFEC). In both definitions, $\alpha = 0$ corresponds to the phase correlation, whereas $\alpha = 1$ corresponds to the cross correlation.

Finding the value of α to achieve an optimal trade-off between noise tolerance and peak sharpness, which may depend on several parameters such as coherence and the number of resolution cells, is beyond the scope of this study; nevertheless, it offers an avenue for enhancing offset estimation precision. For intensity signals, simulation results show that, for $0 \leq \alpha \leq 1$, $\alpha = 1/2$ generally achieves the highest precision under the signal model adopted in this paper.

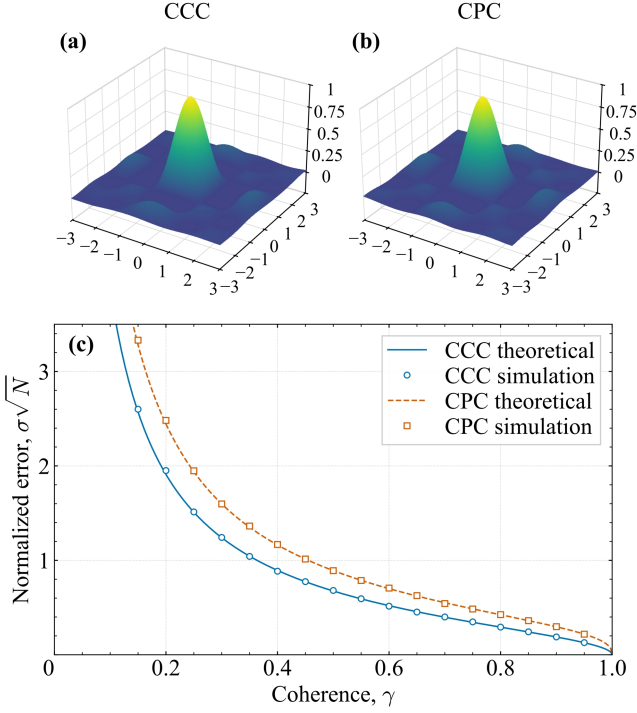


Fig. 1. Comparison of the offset estimation precision between the CCC and CPC under the zero-mean CCG signal model. (a) A CCC correlation surface. (b) A CPC correlation surface. (c) Normalized standard deviation, $\sigma\sqrt{N}$, as a function of coherence γ .

IV. SIMULATION

A. Configuration

We conducted Monte Carlo simulations to validate the derived expressions for offset estimation variances and to compare them with those for cross correlation. Each realization consisted of $N = 4096$ independent, critically sampled complex samples. The coherence γ ranged from 0.15 to 0.95 in increments of 0.05. For each coherence value, 2000 independent realizations were generated.

Each signal pair was generated according to

$$y_n = \gamma x_n + \sqrt{1 - \gamma^2} z_n$$

where x_n and z_n denote independent, unit-power CCG samples. The true offset between x_n and y_n was set to zero. For the correlation of intensity signals, both complex signals were twofold-interpolated by zero-padding their spectra prior to intensity detection. The correlation surfaces were computed in the frequency domain and subsequently transformed into the time domain. The peak position was determined by first finding the integer peak position and subsequently refining it using the Broyden–Fletcher–Goldfarb–Shanno (BFGS) algorithm. The standard deviation was computed from the offset estimates across all realizations. Finally, the normalized standard deviation $\sigma\sqrt{N}$ is reported to ensure independence from N , because all variances given in (13), (14), (28) and (29) scale with $1/N$.

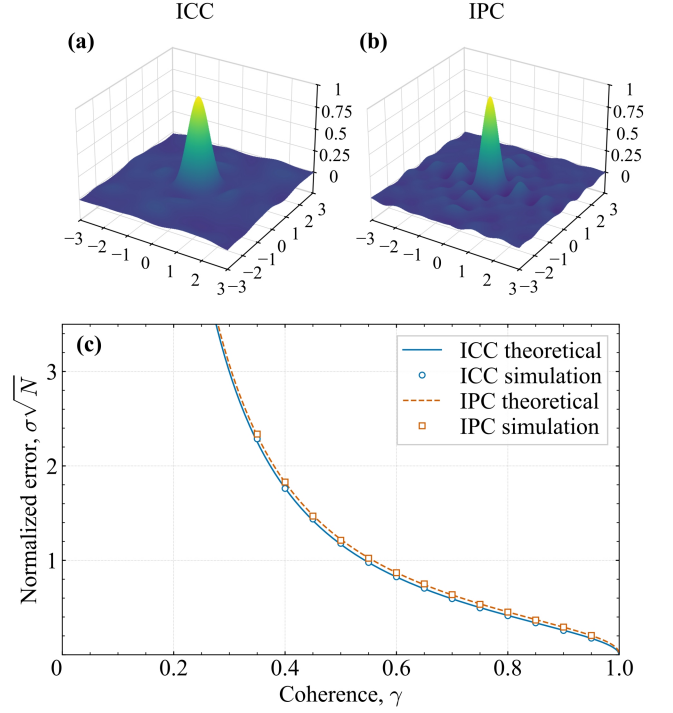


Fig. 2. Comparison of the offset estimation precision between the ICC and IPC under the zero-mean CCG signal model. (a) An ICC correlation surface. (b) An IPC correlation surface. (c) Normalized standard deviation, $\sigma\sqrt{N}$, as a function of coherence γ .

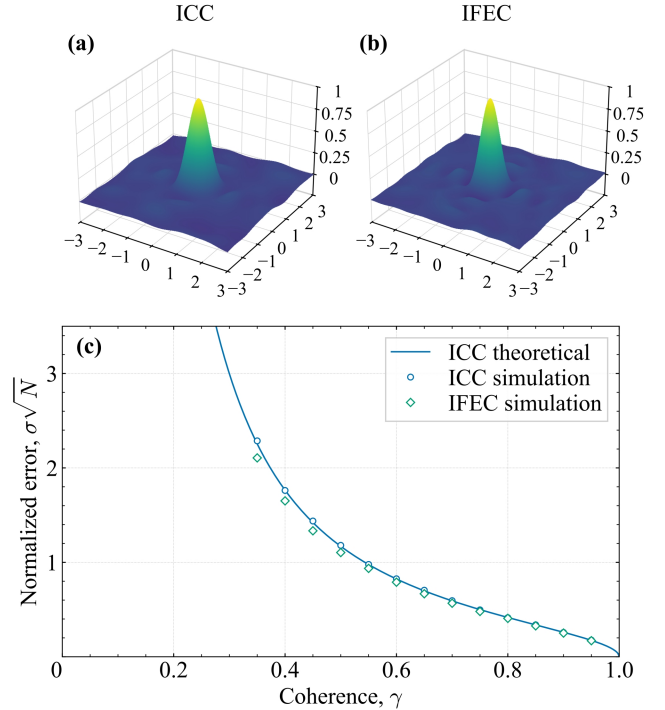


Fig. 3. Comparison of the offset estimation precision between the ICC and IFEC under the zero-mean CCG signal model. (a) An ICC correlation surface. (b) An IFEC correlation surface. (c) Normalized standard deviation, $\sigma\sqrt{N}$, as a function of coherence γ .

B. Result

Fig. 1 compares the results of the CCC and CPC, while Fig. 2 compares those of the ICC and IPC. Generally, the standard deviations are well predicted by (13) and (28). As shown in Fig. 1, although the CPC and CCC correlation peak shapes are highly similar, the normalized-error curves show that the offset estimation precision of the CPC is noticeably lower than that of the CCC. Fig. 2 shows that, although the IPC correlation surface exhibits a sharper peak than the ICC correlation surface, its offset estimation precision is slightly lower. This lower precision of the IPC, despite its sharper peaks, can be explained by the analysis in Section II-D.

Fig. 3 compares the results of the ICC and IFEC, where only the simulation results are shown for the IFEC. By setting $\alpha = 1/2$, the IFEC correlation surface exhibits a peak sharpness intermediate between those of ICC and IPC, while its offset estimation precision is slightly improved compared to the ICC and is the highest among the offset estimators for intensity signals. This improvement generally becomes more pronounced as the coherence decreases.

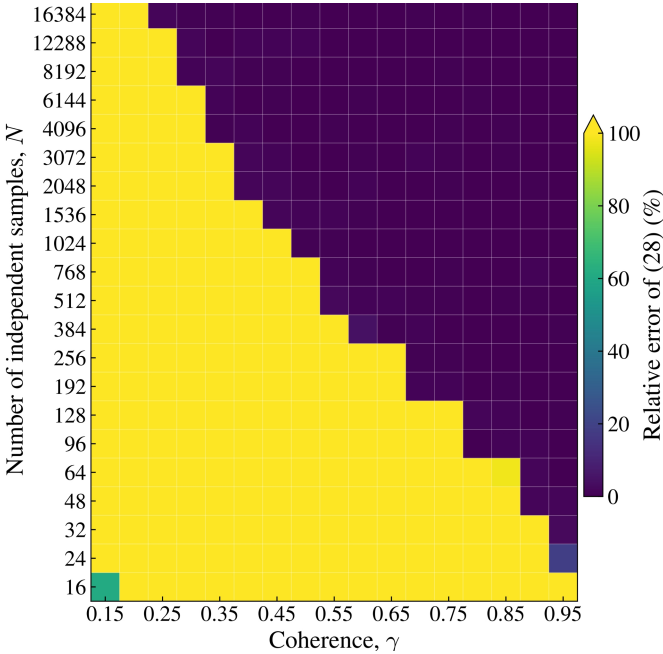


Fig. 4. Relative error of (28) across different numbers of independent samples N and coherence values γ . Each cell is computed from 50000 independent trials.

(28) is derived for large N . We evaluate the relative error of this expression using

$$\varepsilon = \frac{|\hat{\sigma}_{MC}\sqrt{N} - \sigma_{IPC}\sqrt{N}|}{\sigma_{IPC}\sqrt{N}} 100\%$$

The relative error ε generally increases as N decreases. In Fig. 4, the results in the upper triangular region show excellent agreement between the theoretical predictions and the simulation results. However, in the lower triangular

region, the results are corrupted. This is because the approximation via Taylor expansion characterizes local peak position fluctuations under the assumption that the correct mainlobe is selected. For small N and low coherence values, confusion between the mainlobe and sidelobes emerges, as in the ICC [15], and our algorithm cannot accurately pinpoint the mainlobe, leading to outliers and consequently resulting in large discrepancies in the figure.

V. OBSERVATION FROM SAR IMAGERY EXPERIMENT

A. Data and Processing

SAR imagery serves as a typical example of imagery that exhibits speckle. We used two ALOS-2 PALSAR-2 single look complex (SLC) images acquired on March 7, 2016, and April 18, 2016, in stripmap mode over Kumamoto, Japan. The images consist of two consecutive frames, 2950 and 2960, and cover the period encompassing the foreshock and mainshock, which struck on April 14 and 16, 2016.

The images were processed using `alos2App.py` [20] in the ISCE2 software package [21]. The images were first coregistered using geometric offsets [22] computed using the 1-arcsecond (~ 30 m) SRTM DEM [23]. Areas covered by water bodies were masked using the SRTM Water Body Data (SWBD). The ICC, IPC and IFEC with $\alpha = 1/2$ were then performed on the geometrically coregistered images using a 64×64 window with a 32 step to obtain the offsets induced by the earthquakes. For quantitative evaluation, the local standard deviation of the estimated offset was computed using a 9×9 sliding window with a stride of one offset-grid sample.

B. Result

Fig. 5 compares the range offsets estimated by the ICC, IPC, and IFEC. The IPC result contains the largest number of low-precision estimates and isolated outliers. The ICC result is much better. The IFEC produces the highest-precision offset field, with the fewest visible outliers. Table I provides a more detailed quantitative comparison of the local standard deviation among the three methods, which confirms the top performance of IFEC, followed by ICC and then IPC.

TABLE I

Statistical comparison of the local standard deviation of the estimated range offsets. The local standard deviation is computed using a 9×9 window. P95 and P99 denote the 95th and 99th percentiles, respectively.

Method	Mean (pixel)	Median (pixel)	P95 (pixel)	P99 (pixel)	> 1 pixel (%)	> 2 pixels (%)
IPC	0.442	0.044	2.108	3.140	18.92%	5.77%
ICC	0.386	0.048	1.573	2.160	15.82%	1.62%
IFEC	0.221	0.034	1.251	1.916	8.17%	0.81%

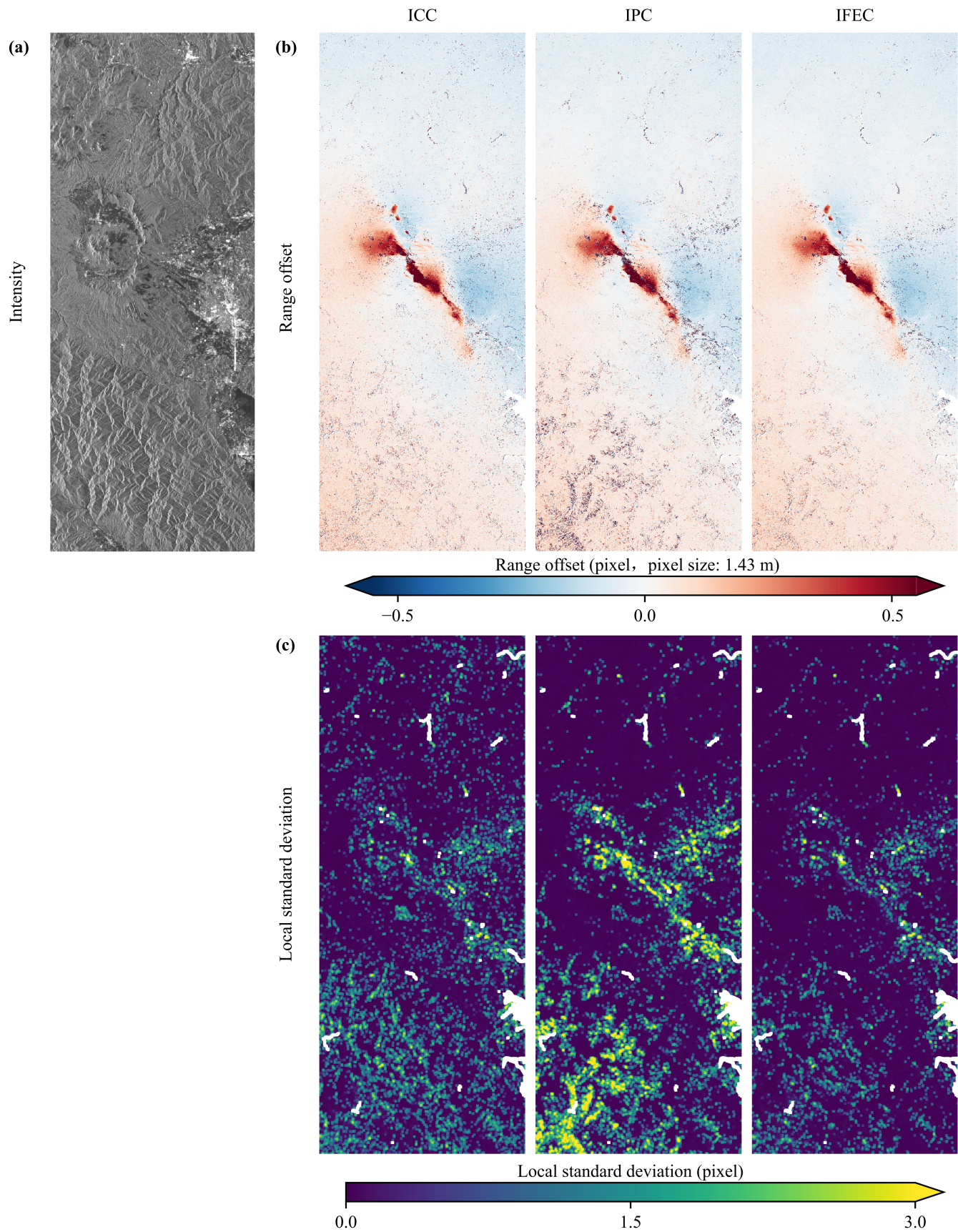


Fig. 5. ICC, IPC, and IFEC offset estimation results for the Kumamoto earthquake. (a) Multilooked intensity image. (b) Estimated range offset. (c) Local standard deviation computed using a 9×9 window.

The SAR images contain not only speckle, but also structural features. Furthermore, a relatively small window size of 64×64 was used. Consequently, these experiments do not constitute a direct, strict validation of the theoretical derivations developed for pure speckle. Nevertheless, the qualitative comparison among the three methods is in agreement with theoretical results. In other words, the empirical performance hierarchy, IFEC > ICC > IPC, remain valid, even in the presence of features under a small window. Therefore, although the theoretical results are derived for pure speckle, they also provide practical guidance in the presence of features.

Compared with the results in Fig. 4, the occurrence of sidelobe confusion significantly drops. This may be attributed to the features present in the images.

VI. CONCLUSION

This paper derives the theoretical offset estimation precision for the CPC and IPC under the zero-mean CCG signal model. The IPC precision is derived under large number of independent samples. It is demonstrated that the CPC exhibits lower precision than the CCC. The IPC similarly yields lower precision than the ICC, but the difference between them is marginal. The mechanism underlying the reduced precision of the CPC and IPC, despite their sharper correlation peaks, is explained. Finally, an avenue for improving offset estimation precision is to tune the exponent of the spectral magnitude of the fractional exponent correlation function.

APPENDIX A PROPERTIES OF $p(\psi; \gamma)$

Since the PDF of the interferometric phase, $p(\psi; \gamma)$, is an even function with respect to ψ and is periodic over the interval $(-\pi, \pi]$, it can be expanded into a Fourier cosine series:

$$p(\psi; \gamma) = \frac{1}{2\pi} \left[1 + 2 \sum_{n=1}^{\infty} C_n \cos(n\psi) \right]. \quad (\text{A.1})$$

Leveraging the orthogonality of the Fourier series, the n -th order Fourier coefficient C_n is exactly equal to the expectation of $\cos(n\psi)$:

$$C_n = E[\cos(n\psi)] = \int_{-\pi}^{\pi} \cos(n\psi) p(\psi; \gamma) d\psi. \quad (\text{A.2})$$

The Fourier coefficients C_n of the interferometric phase for the CCG distribution possess a unified analytical expression in terms of the Gaussian hypergeometric function ${}_2F_1$ [24]:

$$C_n = \frac{\Gamma^2(1 + \frac{n}{2})}{\Gamma(1+n)} \gamma^n {}_2F_1\left(\frac{n}{2}, \frac{n}{2}; n+1; \gamma^2\right). \quad (\text{A.3})$$

A. Derivation of Cosine Expectation $C(\gamma)$

The cosine expectation is exactly the first-order Fourier coefficient. Substituting $n = 1$ into (A.3) and using the specific Gamma function values $\Gamma(3/2) = \sqrt{\pi}/2$ and $\Gamma(2) = 1$, we obtain:

$$\begin{aligned} C(\gamma) = C_1 = E[\cos \psi] &= \frac{\Gamma^2(3/2)}{\Gamma(2)} \gamma {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 2; \gamma^2\right) \\ &= \frac{\pi\gamma}{4} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 2; \gamma^2\right). \end{aligned} \quad (\text{A.4})$$

Using the Gauss contiguous relations for the complete elliptic integrals of the first and second kinds, $K(\gamma^2)$ and $E(\gamma^2)$ [25], the hypergeometric function can be expanded as:

$${}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 2; \gamma^2\right) = \frac{4}{\pi\gamma^2} [E(\gamma^2) - (1 - \gamma^2)K(\gamma^2)]. \quad (\text{A.5})$$

Substituting (A.5) into (A.4) yields the closed-form solution for the cosine expectation:

$$C(\gamma) = \frac{E(\gamma^2) - (1 - \gamma^2)K(\gamma^2)}{\gamma}. \quad (\text{A.6})$$

B. Derivation of Sine Variance $S(\gamma)$

Using the trigonometric identity $\sin^2 \psi = [1 - \cos(2\psi)]/2$, the sine variance $S(\gamma)$ can be determined by the second-order Fourier coefficient:

$$S(\gamma) = E[\sin^2 \psi] = \frac{1}{2} - \frac{1}{2} E[\cos(2\psi)] = \frac{1 - C_2}{2}. \quad (\text{A.7})$$

Substituting $n = 2$ into (A.3) and given that $\Gamma(2) = 1$ and $\Gamma(3) = 2$, the second-order Fourier coefficient is obtained as:

$$C_2 = \frac{1}{2} \gamma^2 {}_2F_1(1, 1; 3; \gamma^2). \quad (\text{A.8})$$

Using the Maclaurin series expansion of the logarithmic function, the hypergeometric function with these specific parameters is strictly equivalent to an algebraic expression containing a logarithmic term [25]:

$$C_2 = 1 + \frac{1 - \gamma^2}{\gamma^2} \ln(1 - \gamma^2). \quad (\text{A.9})$$

Substituting C_2 into (A.7) yields the expression for the sine variance

$$\begin{aligned} S(\gamma) &= \frac{1}{2} \left[1 - \left(1 + \frac{1 - \gamma^2}{\gamma^2} \ln(1 - \gamma^2) \right) \right] \\ &= \frac{\gamma^2 - 1}{2\gamma^2} \ln(1 - \gamma^2). \end{aligned} \quad (\text{A.10})$$

APPENDIX B PROPERTIES OF THE INTENSITY OF THE ZERO-MEAN CCG RANDOM VARIABLE WITH UNITY POWER

Assuming x follows the zero-mean CCG distribution with unity power, i.e., $x \sim \mathcal{CN}(0, 1)$, as defined in Section II-A, the PDF of its intensity, $I = |x|^2$, is given by

$$p(I) = e^{-I}, I \geq 0. \quad (\text{B.1})$$

The expectation of I is $E(I) = E(|x|^2) = 1$. Using the complex Gaussian moment theorem [26], the variance of I can be calculated as

$$\text{Var}(I) = E(I^2) - [E(I)]^2$$

$$\begin{aligned}
&= E(xx^*x^*) - 1 \\
&= E(xx^*)E(xx^*) + E(xx^*)E(xx^*) - 1 \\
&= 1.
\end{aligned} \tag{B.2}$$

The coherence between the reference and secondary intensities, $I_x = |x|^2$ and $I_y = |y|^2$, is given by

$$\begin{aligned}
\gamma_I &= \frac{\text{Cov}(I_x, I_y)}{\sqrt{\text{Var}(I_x)}\sqrt{\text{Var}(I_y)}} \\
&= \text{Cov}(I_x, I_y) \\
&= E(I_x I_y^*) - E(I_x)E(I_y^*) \\
&= E(I_x I_y) - 1 \\
&= E(xy x^* y^*) - 1 \\
&= E(xx^*)E(yy^*) + E(xy^*)E(yx^*) - 1 \\
&= E(xy^*)E(yx^*) \\
&= |\gamma|^2
\end{aligned} \tag{B.3}$$

where the complex Gaussian moment theorem has been used again to derive the sixth line.

APPENDIX C STATISTICAL PROPERTIES OF THE TWOFOLD OVERSAMPLED SIGNALS

A. Coherence between the Intensities in the Frequency Domain

Under the signal model in Section II-A, we have

$$E[X_p] = \sum_{n \in \mathcal{J}_N} E[x_n] e^{-\frac{j2\pi pn}{N}} = 0, E[Y_p] = 0 \tag{C.1a}$$

$$\begin{aligned}
E[X_p X_q^*] &= \sum_{n \in \mathcal{J}_N} \sum_{t \in \mathcal{J}_N} E[x_n x_t^*] e^{-j2\pi(pn - qt)/N} \\
&= \sum_{n \in \mathcal{J}_N} e^{-j2\pi(p - q)n/N} = N \mathbf{1}_{\{p=q\}} \\
E[Y_p Y_q^*] &= E[X_p X_q^*] = N \mathbf{1}_{\{p=q\}}
\end{aligned} \tag{C.1b}$$

and

$$\begin{aligned}
E[X_p Y_q^*] &= \sum_{n \in \mathcal{J}_N} \sum_{t \in \mathcal{J}_N} E[x_n y_t^*] e^{-j2\pi(pn - qt)/N} \\
&= \gamma \sum_{n \in \mathcal{J}_N} e^{-j2\pi(p - q)n/N} = N\gamma \mathbf{1}_{\{p=q\}} \\
E[Y_p X_q^*] &= E[X_p Y_q^*] = N\gamma \mathbf{1}_{\{p=q\}}
\end{aligned} \tag{C.1c}$$

where

$$\mathbf{1}_{\{p=q\}} = \begin{cases} 0, & p \neq q \\ 1, & p = q \end{cases}$$

According to Section II-A, γ is a nonnegative real quantity. The coherence between X_k and Y_k is given by

$$\gamma_k = \frac{E[X_k Y_k^*]}{\sqrt{E[|X_k|^2]} \sqrt{E[|Y_k|^2]}} = \frac{N\gamma}{\sqrt{N} \cdot N} = \gamma. \tag{C.2}$$

The twofold oversampling is performed by zero-padding the original spectrum given in (3), and the scaling factor of the standard $2N$ -point IDFT is $1/(2N)$. Multiplying by an additional factor of 2 restores the original sample values at the even indices. Hence, for $m \in \mathcal{J}_{2N}$,

$$\begin{aligned}
x'_m &= 2 \left(\frac{1}{2N} \sum_{p \in \mathcal{J}_N} X_p e^{j\frac{2\pi pm}{2N}} \right) = \frac{1}{N} \sum_{p \in \mathcal{J}_N} X_p e^{j\frac{2\pi pm}{2N}}, \\
y'_m &= 2 \left(\frac{1}{2N} \sum_{p \in \mathcal{J}_N} Y_p e^{j\frac{2\pi pm}{2N}} \right) = \frac{1}{N} \sum_{p \in \mathcal{J}_N} Y_p e^{j\frac{2\pi pm}{2N}}.
\end{aligned} \tag{C.3}$$

At the even indices, $x'_{2n} = x_n$ and $y'_{2n} = y_n$.

The intensities of the twofold oversampled signals are $u_m = |x'_m|^2$ and $v_m = |y'_m|^2$. Their standard $2N$ -point DFT coefficients are given by

$$\begin{aligned}
U_k &= \sum_{m \in \mathcal{J}_{2N}} u_m e^{-\frac{j2\pi km}{2N}}, \\
V_k &= \sum_{m \in \mathcal{J}_{2N}} v_m e^{-\frac{j2\pi km}{2N}}, k \in \mathcal{J}_{2N}.
\end{aligned} \tag{C.4}$$

Substituting (C.3) into the definition of U_k yields

$$\begin{aligned}
U_k &= \sum_{m \in \mathcal{J}_{2N}} \left(\frac{1}{N^2} \sum_{p \in \mathcal{J}_N} \sum_{q \in \mathcal{J}_N} X_p X_q^* e^{j2\pi(p - q)m/(2N)} \right) e^{-j2\pi km/(2N)} \\
&= \frac{1}{N^2} \sum_{p \in \mathcal{J}_N} \sum_{q \in \mathcal{J}_N} X_p X_q^* \sum_{m \in \mathcal{J}_{2N}} e^{j2\pi(p - q - k)m/(2N)}.
\end{aligned} \tag{C.5}$$

(C.5) can be simplified using the centered $2N$ -point Fourier orthogonality relation given by

$$\sum_{m \in \mathcal{J}_{2N}} e^{j2\pi am/(2N)} = 2N \mathbf{1}_{\{a \equiv 0 \pmod{2N}\}}, a \in \mathbb{Z}. \tag{C.6}$$

Since $p - q \in \{-N + 1, \dots, N - 1\}$, no circular wraparound occurs for $1 \leq k \leq N - 1$. Therefore, the condition in (C.5) reduces to $p - q = k$, yielding

$$\begin{aligned}
U_k &= \frac{2N}{N^2} \sum_{p \in \mathcal{J}_N} \sum_{q \in \mathcal{J}_N} X_p X_q^* \mathbf{1}_{\{p - q = k\}} \\
&= \frac{2}{N} \sum_{r = -N/2 + 1}^{N/2 - k} X_{r+k} X_r^* \\
&= \frac{2}{N} \sum_{r \in \mathcal{R}_k} X_{r+k} X_r^*
\end{aligned} \tag{C.7}$$

where

$$\mathcal{R}_k = \left\{ -\frac{N}{2} + 1, \dots, \frac{N}{2} - k \right\}, 1 \leq k \leq N - 1.$$

When $\mathcal{R}_k$ contains a sufficiently large number of elements, U_k approaches a CCG distribution by virtue of the central limit theorem. At frequency bin $k = N$, $\mathcal{R}_k$ is empty, and $U_N = V_N = 0$. At frequency bin $k = 0$, U_0 and V_0 are the DC components and are proportional to the total intensities of the respective signals. Unlike the nonzero-frequency coefficients considered below, they are real, nonzero-mean quantities and therefore do not constitute a jointly zero-mean CCG pair characterized by the coherence γ_I to be derived. Moreover, the normalized cross-spectrum coefficient at DC is $q_0 = U_0 V_0^* / |U_0 V_0^*| = 1$. Since $\omega_0 = 0$, this term contributes only an offset-independent constant to the correlation function and therefore does not affect the peak position. Accordingly, the coherence derivation is restricted to frequency bins $1 \leq k \leq N - 1$. Since u_m is real, $U_{-k} = U_k^*$.

For $1 \leq k \leq N - 1$, the expectation of U_k is given by

$$E[U_k] = \frac{2}{N} \sum_{r \in \mathcal{R}_k} E[X_{r+k} X_r^*] = \frac{2}{N} \sum_{r \in \mathcal{R}_k} N \mathbf{1}_{\{r+k=r\}} = 0. \quad (\text{C.8})$$

For $1 \leq k \leq N - 1$, the second moment of U_k is obtained by applying the complex Gaussian moment theorem:

$$\begin{aligned} E[|U_k|^2] &= \frac{4}{N^2} \sum_{r \in \mathcal{R}_k} \sum_{s \in \mathcal{R}_k} E[X_{r+k} X_r^* X_{s+k}^* X_s] \\ &= \frac{4}{N^2} \sum_{r \in \mathcal{R}_k} \sum_{s \in \mathcal{R}_k} (E[X_{r+k} X_r^*] E[X_s X_{s+k}^*] \\ &\quad + E[X_{r+k} X_{s+k}^*] E[X_s X_r^*]) \\ &= \frac{4}{N^2} \sum_{r \in \mathcal{R}_k} \sum_{s \in \mathcal{R}_k} (0 + N \mathbf{1}_{\{r=s\}} N \mathbf{1}_{\{s=r\}}) \\ &= 4 \sum_{r \in \mathcal{R}_k} \sum_{s \in \mathcal{R}_k} \mathbf{1}_{\{r=s\}} \\ &= 4 \sum_{r \in \mathcal{R}_k} 1 \\ &= 4(N - k). \end{aligned} \quad (\text{C.9})$$

For V_k , an analogous derivation yields its expression

$$V_k = \frac{2}{N} \sum_{r \in \mathcal{R}_k} Y_{r+k} Y_r^*. \quad (\text{C.10})$$

and second moment

$$E[|V_k|^2] = 4(N - k). \quad (\text{C.11})$$

For $1 \leq k \leq N - 1$, the cross moment between U_k and V_k is evaluated similarly:

$$\begin{aligned} E[U_k V_k^*] &= \frac{4}{N^2} \sum_{r \in \mathcal{R}_k} \sum_{s \in \mathcal{R}_k} E[X_{r+k} X_r^* Y_{s+k}^* Y_s] \\ &= \frac{4}{N^2} \sum_{r \in \mathcal{R}_k} \sum_{s \in \mathcal{R}_k} (E[X_{r+k} X_r^*] E[Y_s Y_{s+k}^*] + E[X_{r+k} Y_{s+k}^*] E[Y_s X_r^*]) \end{aligned}$$

$$\begin{aligned} &= \frac{4}{N^2} \sum_{r \in \mathcal{R}_k} \sum_{s \in \mathcal{R}_k} (0 + N \gamma \mathbf{1}_{\{r=s\}} N \gamma \mathbf{1}_{\{s=r\}}) \\ &= 4 \gamma^2 \sum_{r \in \mathcal{R}_k} \sum_{s \in \mathcal{R}_k} \mathbf{1}_{\{r=s\}} \\ &= 4 \gamma^2 \sum_{r \in \mathcal{R}_k} 1 \\ &= 4(N - k) \gamma^2. \end{aligned} \quad (\text{C.12})$$

Using (C.9), (C.11), and (C.12), the coherence between U_k and V_k is derived as

$$\gamma_{I,k} = \frac{E[U_k V_k^*]}{\sqrt{E[|U_k|^2] E[|V_k|^2]}} = \frac{4(N - k) \gamma^2}{\sqrt{4(N - k) 4(N - k)}} = \gamma^2 \equiv \gamma_I \quad (\text{C.13})$$

where γ_I is defined in Appendix B.

B. Cross-Frequency Expectation $E[\sin \psi_k \sin \psi_l]$

Under the standard DFT, $X_r \sim \mathcal{CN}(0, N)$ and $Y_r \sim \mathcal{CN}(0, N)$. As given in (C.2), the coherence between X_r and Y_r is γ . We write Y_r as

$$Y_r = \gamma X_r + \sqrt{1 - \gamma^2} Z_r, \quad r \in \mathcal{J}_N. \quad (\text{C.14})$$

where $Z_r \sim \mathcal{CN}(0, N)$, and Z_r is independent of X_r . The pairs (X_r, Z_r) are independent for distinct r . Below, we derive the cross-frequency $E[\sin \psi_k \sin \psi_l]$ in three major steps.

First, we define the dimensionless sample-wise phase-noise projection

$$\zeta_r \equiv \frac{\text{Im}(X_r Y_r^*)}{N \gamma} = \frac{1}{N \gamma} \text{Im}(\gamma |X_r|^2 + \sqrt{1 - \gamma^2} X_r Z_r^*). \quad (\text{C.15a})$$

Since $|X_r|^2$ is real,

$$\zeta_r = \frac{\sqrt{1 - \gamma^2}}{N \gamma} \text{Im}(X_r Z_r^*). \quad (\text{C.15b})$$

it follows immediately that

$$E[\zeta_r] = 0. \quad (\text{C.16})$$

Moreover, the properness of $X_r Z_r^*$ yields

$$\begin{aligned} E[\text{Im}^2(X_r Z_r^*)] &= \frac{1}{2} (E[|X_r Z_r^*|^2] - \text{Re}\{E[(X_r Z_r^*)^2]\}) \\ &= \frac{1}{2} (E[|X_r|^2] E[|Z_r|^2] - \text{Re}\{E[X_r^2] E[(Z_r^*)^2]\}) \\ &= \frac{1}{2} (N \cdot N - 0) \\ &= \frac{N^2}{2}. \end{aligned} \quad (\text{C.17})$$

Consequently, for $r, s \in \mathcal{J}_N$,

$$\begin{aligned}
E[\zeta_r \zeta_s] &= \frac{1 - \gamma^2}{N^2 \gamma^2} E[\text{Im}(X_r Z_r^*) \text{Im}(X_s Z_s^*)] \\
&= \frac{1 - \gamma^2}{2\gamma^2} \mathbf{1}_{\{r=s\}} = \frac{1 - \gamma_I}{2\gamma_I} \mathbf{1}_{\{r=s\}}. \quad (\text{C.18})
\end{aligned}$$

Next, we introduce auxiliary deterministic local phase perturbations φ_r via

$$E[X_r Y_r^* | \varphi_r] = N\gamma e^{j\varphi_r}, \quad E[X_r^* Y_r | \varphi_r] = N\gamma e^{-j\varphi_r}. \quad (\text{C.19})$$

Using (C.7) and (C.10), and applying the complex Gaussian moment theorem, we obtain

$$\begin{aligned}
E[U_k V_k^* | \boldsymbol{\varphi}] &= \frac{4}{N^2} \sum_{r \in \mathcal{R}_k} \sum_{s \in \mathcal{R}_k} E[X_{r+k} X_r^* Y_{s+k}^* Y_s | \boldsymbol{\varphi}] \\
&= \frac{4}{N^2} \sum_{r \in \mathcal{R}_k} \sum_{s \in \mathcal{R}_k} E[X_{r+k} X_r^* | \boldsymbol{\varphi}] E[Y_{s+k}^* Y_s | \boldsymbol{\varphi}] \\
&\quad + \frac{4}{N^2} \sum_{r \in \mathcal{R}_k} \sum_{s \in \mathcal{R}_k} E[X_{r+k} Y_{s+k}^* | \boldsymbol{\varphi}] E[Y_s X_r^* | \boldsymbol{\varphi}]. \quad (\text{C.20a})
\end{aligned}$$

For $1 \leq k \leq N-1$, the first term in (C.20a) evaluates to zero. In the second term, distinct DFT bins are independent; hence

$$\begin{aligned}
E[X_{r+k} Y_{s+k}^* | \boldsymbol{\varphi}] &= N\gamma e^{j\varphi_{r+k}} \mathbf{1}_{\{r=s\}}, \\
E[Y_s X_r^* | \boldsymbol{\varphi}] &= N\gamma e^{-j\varphi_r} \mathbf{1}_{\{s=r\}}. \quad (\text{C.20b})
\end{aligned}$$

Substituting (C.20b) into (C.20a) yields

$$\begin{aligned}
E[U_k V_k^* | \boldsymbol{\varphi}] &= \frac{4}{N^2} \sum_{r \in \mathcal{R}_k} \sum_{s \in \mathcal{R}_k} N\gamma e^{j\varphi_{r+k}} \mathbf{1}_{\{r=s\}} N\gamma e^{-j\varphi_r} \mathbf{1}_{\{s=r\}} \\
&= 4\gamma^2 \sum_{r \in \mathcal{R}_k} e^{j(\varphi_{r+k} - \varphi_r)} \\
&= 4\gamma_I \sum_{r \in \mathcal{R}_k} e^{j(\varphi_{r+k} - \varphi_r)}. \quad (\text{C.20c})
\end{aligned}$$

For small φ_r ,

$$\begin{aligned}
E[U_k V_k^* | \boldsymbol{\varphi}] &= 4\gamma_I \sum_{r \in \mathcal{R}_k} [1 + j(\varphi_{r+k} - \varphi_r)] + O(\|\boldsymbol{\varphi}\|^2) \\
&= 4(N-k)\gamma_I [1 + jT_k(\boldsymbol{\varphi})] + O(\|\boldsymbol{\varphi}\|^2) \\
&= 4(N-k)\gamma_I e^{jT_k(\boldsymbol{\varphi})} + O(\|\boldsymbol{\varphi}\|^2) \quad (\text{C.21})
\end{aligned}$$

where the linear phase-response functional is defined as

$$T_k(\boldsymbol{\varphi}) \equiv \frac{1}{N-k} \sum_{r \in \mathcal{R}_k} (\varphi_{r+k} - \varphi_r). \quad (\text{C.22})$$

The random first-order binwise phase perturbation is defined by applying the functional given by (C.22) to the sample-wise projections ζ_r :

$$\vartheta_k \equiv T_k(\boldsymbol{\zeta}) = \frac{1}{N-k} \sum_{r \in \mathcal{R}_k} (\zeta_{r+k} - \zeta_r). \quad (\text{C.23})$$

Reindexing the first sum in (C.23) yields

$$\begin{aligned}
\vartheta_k &= \frac{1}{N-k} \left(\sum_{r=-N/2+1+k}^{N/2} \zeta_r - \sum_{r=-N/2+1}^{N/2-k} \zeta_r \right) \\
&= \frac{1}{N-k} \sum_{r \in \mathcal{J}_N} b_{k,r} \zeta_r \quad (\text{C.24})
\end{aligned}$$

where

$$b_{k,r} \equiv \mathbf{1}_{\{-N/2+1+k \leq r \leq N/2\}} - \mathbf{1}_{\{-N/2+1 \leq r \leq N/2-k\}}. \quad (\text{C.25})$$

For two binwise perturbations, their second-order cross moment is given by

$$\begin{aligned}
E[\vartheta_k \vartheta_l] &= \frac{1}{(N-k)(N-l)} \sum_{r \in \mathcal{J}_N} \sum_{s \in \mathcal{J}_N} b_{k,r} b_{l,s} E[\zeta_r \zeta_s] \\
&= \frac{1 - \gamma_I}{2\gamma_I(N-k)(N-l)} \sum_{r \in \mathcal{J}_N} \sum_{s \in \mathcal{J}_N} b_{k,r} b_{l,s} \mathbf{1}_{\{r=s\}} \\
&= \frac{1 - \gamma_I}{2\gamma_I(N-k)(N-l)} \sum_{r \in \mathcal{J}_N} b_{k,r} b_{l,r} \\
&= \frac{1 - \gamma_I}{\gamma_I} \frac{\min(k, N-l)}{(N-k)(N-l)}. \quad (\text{C.26})
\end{aligned}$$

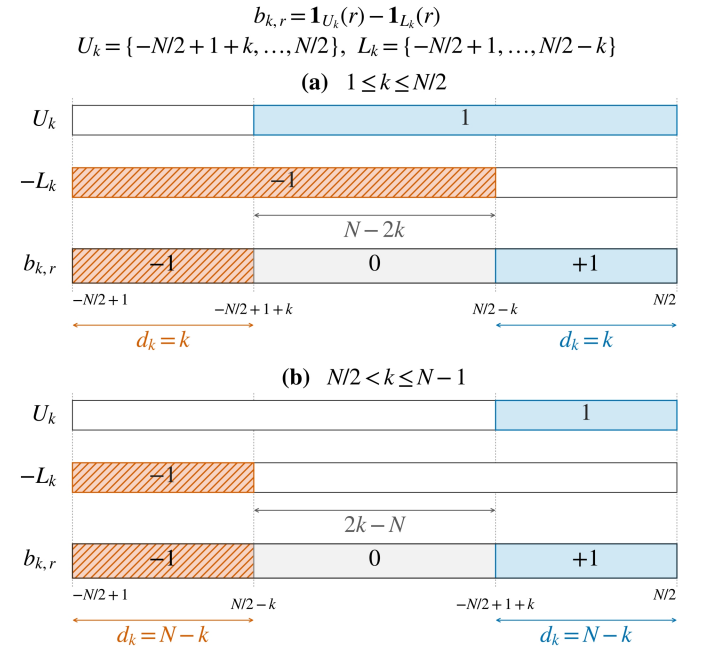


Fig. C.1. $b_{k,r}$ for (a) $1 \leq k \leq N/2$ and (b) $N/2 < k \leq N-1$.

The final result in (C.26) can be obtained by analyzing the coefficients $b_{k,r}$ in (C.25). Fig. C.1 illustrates the two cases regarding the relative positions of the two index sets for the first and second terms in (C.25). In either case, the number of nonzero coefficients at each endpoint is given by

$$d_k \equiv \min(k, N-k). \quad (\text{C.27})$$

Then $b_{k,r}$ can be written as

$$b_{k,r} = \begin{cases} -1, & -N/2+1 \leq r \leq -N/2+d_k, \\ +1, & N/2-d_k+1 \leq r \leq N/2, \\ 0, & \text{otherwise.} \end{cases} \quad (\text{C.28})$$

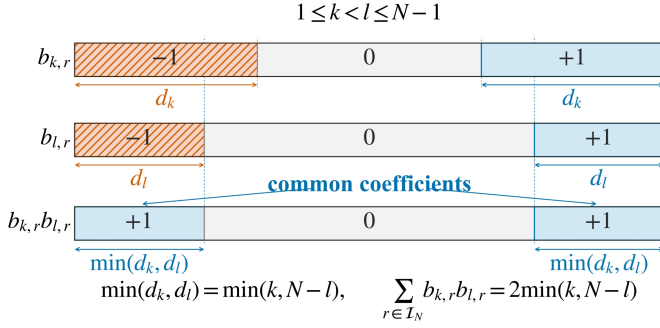


Fig. C.2. Common nonzero coefficients of $b_{k,r}$ and $b_{l,r}$, and the resulting product $b_{k,r}b_{l,r}$.

For $1 \leq k < l \leq N-1$, as illustrated in Fig. C.2, the number of common nonzero coefficients between $b_{k,r}$ and $b_{l,r}$ at either endpoint is $\min(d_k, d_l) = \min(k, N-l)$. In both regions of common nonzero coefficients, each product $b_{k,r}b_{l,r}$ equals 1. Therefore,

$$\sum_{r \in \mathcal{I}_N} b_{k,r} b_{l,r} = 2\min(k, N-l). \quad (\text{C.29})$$

Finally, we use the phase-only intensity cross-spectrum phasor q_k defined in Section II-A. The unperturbed distribution of ψ_k is symmetric; hence, $E[\sin\psi_k] = 0$ and $E[q_k] = E[\cos\psi_k] \equiv \mu_k$ is real. A small phase perturbation ϑ_k rotates the mean phasor:

$$E[q_k|\vartheta_k] = \mu_k e^{j\vartheta_k} + \mathcal{O}(\vartheta_k^2). \quad (\text{C.30})$$

Taking the imaginary part and using $\sin\vartheta_k = \vartheta_k + \mathcal{O}(\vartheta_k^3)$ yields

$$E[\sin\psi_k|\vartheta_k] = \mu_k \sin\vartheta_k = \mu_k \vartheta_k + \mathcal{O}(\vartheta_k^3). \quad (\text{C.31})$$

For $k \neq l$,

$$\begin{aligned} E[\sin\psi_k \sin\psi_l] &= E[(\sin\psi_k|\vartheta_k + \epsilon_k)(\sin\psi_l|\vartheta_l + \epsilon_l)] \\ &= E[\sin\psi_k|\vartheta_k \cdot \sin\psi_l|\vartheta_l] + E[\sin\psi_l|\vartheta_l \cdot \epsilon_k] \\ &\quad + E[\sin\psi_k|\vartheta_k \cdot \epsilon_l] + E[\epsilon_k \cdot \epsilon_l] \\ &= E[\mu_k \vartheta_k + \mathcal{O}(\vartheta_k^3)][\mu_l \vartheta_l + \mathcal{O}(\vartheta_l^3)] + E[\sin\psi_l|\vartheta_l \cdot \epsilon_k] \\ &\quad + E[\sin\psi_k|\vartheta_k \cdot \epsilon_l] + E[\epsilon_k \cdot \epsilon_l] \\ &= E[\mu_k \mu_l \vartheta_k \vartheta_l] + E[\mu_k \vartheta_k \mathcal{O}(\vartheta_l^3)] + E[\mu_l \vartheta_l \mathcal{O}(\vartheta_k^3)] \\ &\quad + E[\mathcal{O}(\vartheta_l^3) \mathcal{O}(\vartheta_k^3)] + E[\sin\psi_l|\vartheta_l \cdot \epsilon_k] \\ &\quad + E[\sin\psi_k|\vartheta_k \cdot \epsilon_l] + E[\epsilon_k \cdot \epsilon_l] \end{aligned} \quad (\text{C.32a})$$

Neglecting the higher-order residual terms yields

$$E[\sin\psi_k \sin\psi_l] \simeq E[\mu_k \mu_l \vartheta_k \vartheta_l]. \quad (\text{C.32b})$$

For sufficiently large $(N-k)$ and $(N-l)$, the pairs (U_k, V_k) and (U_l, V_l) asymptotically approach jointly zero-mean CCG pairs with coherence γ_l ; hence,

$$\mu_k \simeq \mu_l \simeq C(\gamma_l). \quad (\text{C.33})$$

Substituting (C.26) and (C.33) into (C.32b) yields

$$\begin{aligned} E[\sin\psi_k \sin\psi_l] &\simeq C^2(\gamma_l) E[\vartheta_k \vartheta_l] \\ &= C^2(\gamma_l) \frac{1 - \gamma_l}{\gamma_l} \frac{\min(k, N-l)}{(N-k)(N-l)} \end{aligned} \quad (\text{C.34})$$

where $1 \leq k < l \leq N-1$. (C.34) is accurate for large N .

APPENDIX D QUANTITATIVE COMPARISON BETWEEN ICC AND IPC

The ICC and IPC differ significantly in both the peak curvature expectation and the peak position variance of their correlation surfaces, precluding a straightforward direct comparison. Therefore, we normalize their correlation surfaces prior to comparison.

A. Peak Position Variance and Peak Curvature Expectation of the Normalized ICC Correlation Surface

Let $c_{\text{ICC}}(\xi)$ denote the ICC correlation surface, where ξ is in the original coordinate system before oversampling. Under the homogeneous CCG model, $E[c_{\text{ICC}}(\xi)]$ is derived as [14]

$$E[c_{\text{ICC}}(\xi)] = 2N[1 + \gamma^2 \text{sinc}^2(\xi)]. \quad (\text{D.1})$$

The first term, $2N$, is independent of ξ and hence vanishes in both the first and second derivatives of $c_{\text{ICC}}(\xi)$. Moreover, a normalizing factor for $c_{\text{ICC}}(\xi)$ appears in both the numerator and denominator of the offset estimation variance given in (17). Without changing the offset estimation variance, we subtract the first term, $2N$, from $c_{\text{ICC}}(\xi)$ and normalize it by the signal-dependent peak height, $2N\gamma^2$, to obtain a normalized correlation surface

$$\tilde{c}_{\text{ICC}}(\xi) = \frac{c_{\text{ICC}}(\xi) - 2N}{2N\gamma^2}. \quad (\text{D.2})$$

The expectation of $\tilde{c}_{\text{ICC}}(\xi)$ is given by

$$E[\tilde{c}_{\text{ICC}}(\xi)] = \text{sinc}^2(\xi) \quad (\text{D.3})$$

where $\text{sinc}(\xi) = \sin(\pi\xi)/\pi\xi$, using the Maclaurin expansion

$$\text{sinc}(\xi) = 1 - \frac{\pi^2 \xi^2}{6} + \mathcal{O}(\xi^4). \quad (\text{D.4})$$

we obtain

$$\text{sinc}^2(\xi) = 1 - \frac{\pi^2 \xi^2}{3} + \mathcal{O}(\xi^4). \quad (\text{D.5})$$

Then, the peak curvature (sharpness) expectation is given by

$$\tilde{\kappa}_{\text{ICC}} \equiv |E[\tilde{c}_{\text{ICC}}''(0)]| = \frac{2\pi^2}{3}. \quad (\text{D.6})$$

For large N , the variance of the estimated offset for ICC is derived as [15]

$$\sigma_{\text{ICC}}^2 = \frac{3}{10\pi^2 N} \frac{2 + 5\gamma^2 - 7\gamma^4}{\gamma^4}. \quad (\text{D.7})$$

Combining (17), (D.6) and (D.7) yields the peak position variance (fluctuation)

$$\tilde{\mathcal{V}}_{\text{ICC}} \equiv \text{Var}[\tilde{c}_{\text{ICC}}'(0)] = \frac{2\pi^2}{15N} \frac{2 + 5\gamma^2 - 7\gamma^4}{\gamma^4}. \quad (\text{D.8})$$

B. Peak Position Variance and Peak Curvature Expectation of the Normalized IPC Correlation Surface

Since the intensity is obtained by twofold oversampling of the original complex signals, the offset is doubled in the oversampled coordinate system. Here we set $\tilde{\xi} = 2\xi$, where ξ denotes the offset in the original coordinate system.

The IPC correlation surface can be written as

$$c_{\text{IPC}}(\xi) = \frac{1}{2N} \sum_{k=-N+1}^{N-1} q_k e^{j\omega_k \tilde{\xi}} = \frac{1}{2N} \sum_{k=-N+1}^{N-1} q_k e^{j\omega_k 2\xi} \quad (\text{D.9})$$

where $\omega_k = \pi k/N$, and the phase-only intensity cross-spectrum phasor, q_k , at frequency bin k is defined in Section II-A. The frequency bin at $k = N$ is omitted, as both U_k and V_k result from the convolution of the zero-padded spectra and evaluate to 0 at $k = N$.

For large N , q_k is approximately independent of k , except near a vanishing fraction of edge bins. Its expectation is

$$\mu = E[q_k] = E[\cos\psi_k]. \quad (\text{D.10})$$

Then $E[c_{\text{IPC}}(\xi)]$ can be approximated as

$$E[c_{\text{IPC}}(\xi)] \simeq \frac{\mu}{2N} \sum_{k=-N+1}^{N-1} e^{j\omega_k \tilde{\xi}}. \quad (\text{D.11})$$

Again, without changing the offset estimation variance, we normalize $c_{\text{IPC}}(\xi)$ by its expectation at $\xi = 0$ to obtain a normalized correlation surface

$$\tilde{c}_{\text{IPC}}(\xi) = \frac{c_{\text{IPC}}(\xi)}{E[c_{\text{IPC}}(0)]}. \quad (\text{D.12})$$

Its expectation converges to a Riemann integral:

$$\begin{aligned} E[\tilde{c}_{\text{IPC}}(\xi)] &\equiv E\left[\frac{c_{\text{IPC}}(\xi)}{E[c_{\text{IPC}}(0)]}\right] \\ &\simeq \frac{1}{2} \int_{-1}^1 e^{j2\pi f \xi} df \\ &= \text{sinc}(2\xi) \end{aligned} \quad (\text{D.13})$$

where $f = k/N$. Using the Maclaurin expansion again

$$\text{sinc}(2\xi) = 1 - \frac{2\pi^2 \xi^2}{3} + \mathcal{O}(\xi^4). \quad (\text{D.14})$$

the peak curvature (sharpness) expectation is derived as

$$\tilde{\kappa}_{\text{IPC}} \equiv |E[\tilde{c}_{\text{IPC}}''(0)]| = \frac{4\pi^2}{3}. \quad (\text{D.15})$$

Combining (D.15) and (28) yields the peak position variance

$$\tilde{V}_{\text{IPC}} \equiv \text{Var}[\tilde{c}_{\text{IPC}}'(0)] = \frac{2\pi^2}{3N} B \quad (\text{D.16})$$

where

$$B = \frac{(\gamma^4 - 1) \ln(1 - \gamma^4)}{[E(\gamma^4) - (1 - \gamma^4)K(\gamma^4)]^2} + (\pi^2 - 5) \frac{1 - \gamma^2}{\gamma^2}$$

The first and second terms of B denote the co-frequency and cross-frequency contributions, respectively.

As validation of the theory, the central slices of the ensemble-averaged correlation surfaces from Monte Carlo simulations agree closely with (D.3) and (D.13), as shown in Fig. D.1.

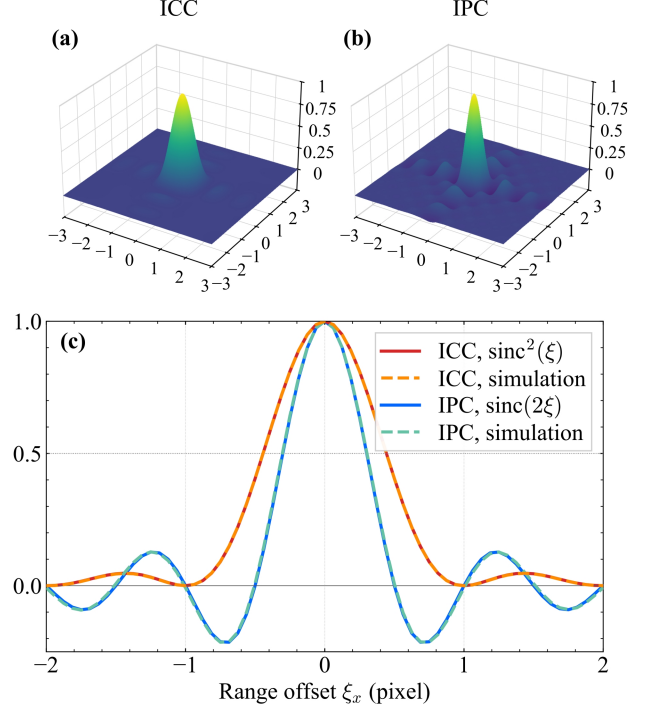


Fig. D.1. Comparison of the correlation surfaces of the ICC and IPC. (a) Ensemble-averaged ICC correlation surface. (b) Ensemble-averaged IPC correlation surface. (c) Central slices.

C. Quantitative Comparison between ICC and IPC

Combining (D.6) and (D.15) yields

$$\frac{\tilde{\kappa}_{\text{IPC}}}{\tilde{\kappa}_{\text{ICC}}} = 2, \quad \frac{\tilde{\kappa}_{\text{IPC}}^2}{\tilde{\kappa}_{\text{ICC}}^2} = 4. \quad (\text{D.17})$$

which quantifies the ratio of peak curvature expectations between the ICC and IPC correlation surfaces.

Similarly, combining (D.8) and (D.16) leads to

$$\frac{\tilde{V}_{\text{IPC}}}{\tilde{V}_{\text{ICC}}} = 5 \frac{\gamma^4 B}{2 + 5\gamma^2 - 7\gamma^4} \quad (\text{D.18})$$

providing a quantitative comparison of their peak position variances.

Finally, the ratio of the offset estimation variances between ICC and IPC is given by

$$\frac{\sigma_{\text{IPC}}^2}{\sigma_{\text{ICC}}^2} = \frac{\tilde{\kappa}_{\text{ICC}}^2 \tilde{V}_{\text{IPC}}}{\tilde{\kappa}_{\text{IPC}}^2 \tilde{V}_{\text{ICC}}} = \frac{1}{4} \frac{\tilde{V}_{\text{IPC}}}{\tilde{V}_{\text{ICC}}}. \quad (\text{D.19})$$

Here we provide numerical results for these comparisons at $\gamma = 0.8$. Table D.I lists the corresponding results.

TABLE D.I

Numerical comparisons between ICC and IPC at $\gamma = 0.8$.

Quantity	ICC	IPC
Peak curvature expectation ($\tilde{\kappa}$)	6.579736	13.159473
Squared peak curvature expectation ($\tilde{\kappa}^2$)	43.29293	173.17172
Peak position variance ($\tilde{V}$)	7.494731/ N	35.583057/ N
Offset estimation standard deviation (σ)	0.416073/ $\sqrt{N}$	0.453297/ $\sqrt{N}$
Offset estimation variance (σ^2)	0.173117/ N	0.205478/ N

The ratio of peak position variances is

$$\frac{\tilde{V}_{\text{IPC}}}{\tilde{V}_{\text{ICC}}} = \frac{35.583057}{7.494731} = 4.747743. \quad (\text{D. 20})$$

whereas the ratio of squared peak curvature expectations is exactly 4. Hence, the ratio of the offset estimation variances is

$$\frac{\sigma_{\text{IPC}}^2}{\sigma_{\text{ICC}}^2} = \frac{4.747743}{4} = 1.186936 \approx 1.187. \quad (\text{D. 21})$$

Therefore, although the IPC exhibits a sharper peak, its larger peak position fluctuation outweighs this sharpness advantage, ultimately leading to lower offset estimation precision.

D. Scope of the Approximation

The above results are derived under the homogeneous CCG model. (D.13) is derived by assuming a large N and an approximately frequency-independent $E[q_k]$. The deviation increases toward the spectral edges due to the decreasing number of samples involved in the convolution used to obtain U_k and V_k for computing q_k . Moreover, both offset estimation variances, σ_{ICC}^2 and σ_{IPC}^2 , are derived within the mainlobes of the $\text{sinc}(\cdot)$ functions given in (D.3) and (D.13). For low coherence or small N , these derivations may become invalid as the mainlobe and sidelobes become indistinguishable, as also discussed for the ICC in [15].

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